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Generalized t-Pebbling Numbers of Wheel and Complete r-partite graph

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Abstract : The generalized t-pebbling number of a graph G, $f_{glt}(G)$, is the least positive integer n such that however n pebbles are placed on the vertices of G, we can move t-pebbles to any vertex by a sequence of moves, each move taking p pebbles off one vertex and placing one on an adjacent vertex. In this paper, we determine the generalized t-pebbling number of wheel W_n and complete r-partite graph.

Key Words : Graph, wheel and complete r-partitle graph.

1 Introduction

Let G be a simple connected graph. The pebbling number of G is the smallest number f(G) such that however these f(G) pebbles are placed on the vertices of G, we can move a pebble to any vertex by a sequence of moves, each move taking two pebbles off one vertex and placing one on an adjacent vertex [2]. Suppose n pebbles are distributed on to the vertices of a graph G, a generalized p pebbling step [u,v] consists of removing p pebbles from a vertex u, and then placing one pebble on an adjacent vertex v, for any $p \geq 2$. Is it possible to move a pebble to a root vertex r, if we can repeatedly apply generalized p pebbling steps? It is answered in the affirmative by Chung in [1]. The **generalized pebbling number** of a vertex v in a graph G is the smallest number $f_{gl}(v,G)$ with the property that from every placement of $f_{gl}(v,G)$ pebbles on G, it is possible to move a pebble to v by a sequence of pebbling move consists of removing p pebbles from a vertex and placing one pebble on an adjacent vertex. The generalized pebbling number of the graph G, denoted by $f_{gl}(G)$, is the maximum $f_{gl}(G)$ over all vertices v in G.

Again the generalized t-pebbling number of a vertex v in a graph G is the smallest number $f_{glt}(v,G)$ with the property that from every placement of $f_{glt}(v,G\}$ pebbles on G, it is possible to move t pebbles to v by a sequence of pebbling moves where a pebbling move consists of the removal of p pebbles from a vertex and the placement of one of these pebbles on an adjacent vertex. The **generalized t-pebbling number** of the graph G, denoted by $f_{glt}(G)$ is the maximum $f_{glt}(v,G)$ over all vertices v of G. Throughout this paper G denotes a simple connected graph with vertex set V(G) and edge set E(G).

$\lfloor x \rfloor$ denote the largest integer less than or equal to x and $\lceil x \rceil$ denote the smallest integer greater than or equal to x.

2 Known Results

We find the following results with regard to the generalized pebbling numbers of graph in [2, 6] and their generalized t-pebbling numbers in [3].

Theorem 2.1. For a complete graph K_n, $f_{gl}(K_n) = (p-1)n-(p-2)$ where $p \geq 2$.

Theorem 2.2. For a path of length n, $f_{gl}(P_n) = p^n$ where $p \geq 2$.

Theorem 2.3. For a star $K_{1,n}$, $f_{gl}(K_{1,n}) = (p-1)n+(p^2-2p+2)$ if $n > 1$ and $p \geq 2$.

Theorem 2.4. The generalized t-pebbling number for a path of length n is $f_{glt}(P_n)=tp^n$.

Theorem 2.5. The generalized t-pebbling number of a complete graph on n vertices where $n \geq 3$, $p \geq 2$ is $f_{glt}(K_n) = pt+(p-1)(n-2)$.

Theorem 2.6. The generalized t-pebbling number for a star $K_{1,n}$ where $n > 1$ is $f_{glt}(K_{1,n})=p^2t+(p-1)(n-2)$ where $p \geq 2$.

Theorem 2.7. For $n \geq 4$, the generalized pebbling number of the wheel graph W_n is $f_{gl}(W_n) = (p-1)+(p^2-2p+1)$ where $p \geq 2$.

Theorem 2.8. The generalized pebbling number of the fan graph F_n is $f_{gl}(F_n) = (p-1)n+(p^2-2p+1)$.

Theorem 2.9. For $G = K_{s_1,s_2,\ldots,s_r}$ the generalized pebbling number is given by

$$f_{gl}(G) = \begin{cases} p^2+(p-1)(s_1-2) & \text{if } p \geq n-s_1 \\ p+(p-1)(n-2) & \text{if } p < n-s_1 \end{cases}.$$

We will now proceed to compute the genearlized t-pebbling numbers of wheel W_n and complete r-partite graph.

3 Computation of Genearlized t-pebbling number

Definition 3.1. We define the wheel graph denoted by W_n to be the graph with $V(W_n)= \{h,v_1,v_2, \ldots ,v_n\}$ where h is called the hub of W_n and $E(W_n)=E(C_n) \cup \{hv_1, hv_2, \ldots , hv_n\}$ where C_n denotes the cycle graph on n vertices.

Theorem 3.2. Let $K_1 = \{h\}$. Let $C_n = \{v_1, v_2, \ldots, v_n\}$ be a cycle of length n. Then the generalized t-pebbling number of the wheel graph W_n is $f_{glt}(W_n) = p^2(t-1)+(p-1)n+(p^2-2p+1)$.

Proof : By Theorem 2.5, $f_{glt}(h,W_n) = pt+(p-1)(n-1)$. Let us now find the generalized t-pebbling number of v_1. Assume that v_1 has zero pebbles. Let us place (p^2t-1) pebbles at $v_{\lceil \frac{n}{2} \rceil}$, (p-2) pebbles at v_n and (p-1) pebbles at each of $w_n \backslash \{v_1, v_{\lceil \frac{n}{2} \rceil}, v_n\}$.

Then t pebbles cannot be moved to v_1.

So $f_{glt}(v_1,W_n) \geq p^2(t-1)+(p-1)n+(p^2-2p+1)$.

Let us use induction on t to prove the $f_{glt}(v_1,W_n) \leq p^2(t-1)+(p-1)n+(p^2-2p+1)$.

For t=1, the result is true by Theorem 2.7.

By distributing $p^2(m-2)+(p-1)n+(p^2-2p+1)$ pebbles on $W_n \backslash \{v_1\}$, then we can move (m-1) pebbles to the target vertex v_1.

That is, $f_{gl(m-1)}(W_n)=p^2(m-2)+(p-1)n+(p^2-2p+1)$. Suppose $p^2(m-1)+(p-1)n+(p^2-2p+1)$ pebbles are distributed on to the vertices of $W_n \backslash \{v_1\}$. Let the target vertex be v_1 of C_n.

If there is a vertex in C_n with at least p^2 pebbles, then a pebble can be moved to v_1. Using only p^2 pebbles through h. The remaining $p^2(m-2)+(p-1)n+(p^2-2p+1)$ pebbles are sufficient to put (m-1) additional pebbles on v_1 by using induction. Otherwise any one of the vertices of $W_n \backslash \{v_1\}$ say $v_{\lceil \frac{n}{2} \rceil}$ receive at least p pebbles and each of the vertices $W_n \backslash \{v_1, v_{\lceil \frac{n}{2} \rceil}\}$ receive p-1 pebbles then from $v_{\lceil \frac{n}{2} \rceil}$ using a sequence of pebbling moves, $v_{\lceil \frac{n}{2} \rceil}$, $v_{\lceil \frac{n}{2} \rceil -1}$, … ,$v_1$ we can move a pebble to v_1. Remaining p^2+(p-1)(n-$\left\lceil \frac{n}{2} \right\rceil$+2)+($p^2$-3p+1) > 0. So by induction, (m-1) pebbles can be moved to v_1. Hence in all cases $f_{glm}(v_1,W_n) \leq p^2(m-1)+(p-1)n+(p^2-2p+1)$. Therefore $f_{glt}(W_n)=p^2(m-1)+(p-1)n+(p^2-2p+1)$.

Definition 3.3. A graph G = (V,E) is called an r-partite graph if V can be partitioned into r non-empty subsets $V_1,V_2, \ldots ,V_r$ such that no edge of G joins vertices in the same set. The sets $V_1,V_2, \ldots ,V_r$ are called partite sets or vertex classes of G. If G is an r-partite graph having partite sets $V_1,V_2, \ldots ,V_r$ such that every vertex of V_i is joined to every vertex of V_j where $1 \leq i, j \leq r$ and $i \neq j$, then G is called a complete r-partite graph. If $|V_i|=s_i$ for i=1,2, … , r then we denote G by $K_{s_1,s_2,\ldots,s_r}$.

Notation 3.4. For $s_1 \geq s_2 \geq \ldots \geq s_r$, $s_1 > 1$ and if r = 2, $s_2 > 1$, let $K_{s_1,s_2,\ldots,s_r}$ be the complete r-partitle graph with s_1, s_2, … , s_r vertices in vertex classes C_1, C_2, … , C_r respectively. Let $n = \sum_{i=1}^{r} s_i$.

Theorem 3.5. For G = $K_{s_1,s_2,\ldots,s_r}$ the generalized t-pebbling number for a complete r-partite graph G is given by

$$f_{glt}(G) = \begin{cases} pt+(p-1)(n-2) & if \;\; pt < n-s_1 \\ p^2 t+(p-1)(s_1-2) & if \;\; pt \geq n-s_1 \end{cases}.$$

Proof :

Case i: Assume $pt < n - s_1$.

Let us place pt+(p-1)(n-2)-1 pebbles on the vertices of G-{v} as follows. Let us choose (t-1) vertices and we place p+(p-1) pebbles on each of the (t-1) vertices and we place (p-1) pebbles each on the remaining vertices clearly t pebbles cannot be moved to v.

Hence $f_{glt}(v,G) > (t-1)[(p+(p-1)]+(p-1)(n-t)$

$$= pt+(p-1)(n-2)-1$$

$$\geq pt+(p-1)(n-2).$$

Next we will use induction to show that pt+(p-1)(n-2) pebbles are sufficient to move t pebbles to any desired vertex. For t=1 results is true by Theorem 2.9. Suppose t > s_1, and pt+(p-1)(n-2) pebbles are placed on the vertices of G. Let the target vertex be v of C_k for some k=1, 2, … , n. If there is a vertex w of C_j ($j \neq k$) with at least p pebbles then a pebble can be placed on v.

The remaining p(t-1)+(p-1)(n-2) pebbles are sufficient to put (t-1) additional pebbles on v by induction. If not then every vertex of $G\backslash C_k$ wil have at most (p-1) pebbles on it. Suppose among these n-s_k vertices, q is the number of vertices with at least one pebble. Therefore there will be pt+(p-1)(n-2)-q pebbles on the vertices of C_k. We consider the following cases.

Subcase I : $q \geq t$.

We use pebbling move from s_k-1 vertices of $C_k\backslash\{v\}$ to put the remaining at most (p-1) pebbles on each of the t of the q occupied vertices of v(G)-C_k. Using (p-1)t pebbles we can pebble t vertices with (p-1) pebbles. Then remaining (p-1)(n-2)-(q-t) pebbles are in $C_k\backslash\{v\}$. From the t vertices with p pebbles we can move t pebbles to v.

Subcase ii : $q < t$.

As in subcase (i) first we will put (p-1) more pebbles on each of these q vertices by maiing (p-1)q moves from the vertices of $C_k \setminus \{v\}$ in order to put q pebbles on v. Then we have to place t-q additional pebbles on v. So we use $p^2(t-q)+(p-1)pq=p^2t-pq$ pebbles among $pt+(p-1)(n-2)-q$ pebbles in the vertices of $C_k \setminus \{v\}$. Hence in all the cases $f_{glt}(v,G) \leq pt+(p-1)(n-2)$.

Case ii: Assume $pt \geq n - s_1$.

Let the vertices of C_1 be $v_1, v_2, \dots, v_n$ and let v_{s_1} be the target vertex. Let us place $p^2t+(p-1)(s\_1-2)$ pebbles on the vertices of C_1 as follows. Let us place p^2t-1 pebbles on v_1 and place (p-1) pebbles each on (s_1-2) vertices of C_1 other than v_1 and v_{s_1}. In this case t-pebbles cannot be moved to v_{s_1}. Hence $f_{glt}(G) \geq p^2t+(p-1)(s_1-2)$.

Next we will use induction on t to prove that $p^2t+(p-1)(s_1-2)$ pebbles are sufficient to put t pebbles on any desired vertex clearly the claim is true for $pt=n-s_1$.

Since by case(i) $f_{glt}(G) = pt+(p-1)(n-2)$

$$= pt+(p-1)(pt+s_1-2)$$

$$= p^2t+(p-1)(s_1-2).$$

Suppose $p(m-1) > n-s_1$ and $f_{gl(m-1)}(G) = p^2t(m-1)+(p-1)(s_1-2) = p^2m+(p-1)s_1-(p^2+2p+2)$.

We prove the result is true for m where $pm > n-s_1$. Suppoe $p^2m+(p-1)(s_1-2)$ pebbles are distributed on the vertices of G. Let the target vertex be v of C_k. If there is a vertex in some C_j ($j \neq k$) with at least p pebbles, then a pebble can be placed on v using only p pebbles. The remaining $p^2m+(p-1)s_1-3p+2$ pebbles are sufficient to put (m-1) additional pebbles on v, since $p^2+2p-2-3p+2 > 0$. If not then every vertex of $G \backslash C_k$ will contain either zero or at least one pebble on it. If there is a vertex say w in some C_j ($j \neq k$) with at least one pebble on it, we use (p-1)p pebbles from the vertices of C_k to put (p-1) pebbles on w and hence a pebble can be placed on v. Since $p^2+2p-2-(p-1)(p+3) > 0$, then remaining $f_{gl(m-1)}(G)$ pebbles would suffice to put (m-1) additional pebbles on v. Otherwise, every vertex of $G \backslash C_k$ will have zero pebbles, using p^2 pebbles we can place a pebble on v in this case the remaining $p^2(m-1)+(p-1)(s_1-2)$ pebbles would suffice to put (m-1) additional pebbles on v. Thus $f_{glm}(v,G) \leq p^2m+(p-1)(s_1-2)$. Therefore by induction $f_{glt}(v,G) \leq p^2t+(p-1)(s_1-2)$ for all $pt < n-s_1$. Thus $f_{glt}(G) < p^2t+(p-1)(s_1-2)$ for all $pt \geq n-s_1$ and so the proof is over.

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